UiO : Department of Mathematics
University of Oslo

Hyperbolic Structure of Knot Complements

János Apró

June 16, 2025

Overview

- Hyperbolic geometry as taught by William Thurston in the 1980s
- (G, X)-manifolds
- Knots and Knot complements
- The theory of gluing polyhedra
- Mondern day tools (SnapPy, Regina)

Hyperbolic Geometry

The most intuitive way to define hyperbolic n-space is through sectional curvatuer, via the Riemannian curvature tensor.

Definition

The sectional curvature denoted $K(\sigma_p)$ or $K(u, v)$ is defined in a two-dimensional linear subspace of the tangent space T_pM spanned by the linearly independent tangent vectors u, v:

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}.$$

The simply connected complete Riemannian n-manifold with constant sectional curvature -1 is called hyperbolic n-space, denoted $\mathbb{H}^n$.

Hyperbolic manifolds

Roughly speaking, a Hyperbolic n-manifold is a manifoldlocally modeled on $\mathbb{H}^n$.

This can be made sense of in the frame of Riemannian Geometry, but for the sake of This thesis we generalize a bit and introduce (G, X)-manifolds.

(G, X)-manifolds

Definition

Let G be a group of similarities of an n-dimensional geometric space X and let M be an n-manifold. A (G, X)-atlas for M is a colletion of charts $\Phi = \{\varphi_i : U_i \to X\}_{i \in \mathcal{I}}$, satisfying:

1. The coordinate neighborhoods U_i are open connected subsets of M for each i.
2. The chart φ_i maps U_i homeomorphically onto an open subset of X for each i.
3. The coordinate neighborhoods $\{U_i\}_{i \in \mathcal{I}}$ cover M.
4. If $U_i \cap U_j \neq \varnothing$ then

$$\varphi_j \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

called a coordinate change, agrees with an element of G in a neighborhood of each point of $U_i \cap U_j$.

(G, X)-manifolds

The following two are the two most notable examples regarding the thesis:

1. A manifold equipped with an $(I(\mathbb{H}^n), \mathbb{H}^n)$-structure is called a *hyperbolic n-manifold*, where $I(\mathbb{H}^n)$ is the group of isometries of $\mathbb{H}^n$.
2. An $(S(\mathbb{E}^n), \mathbb{E}^n)$-structure on a manifold is called a *Euclidean similarity structure*, and an $(S(\mathbb{E}^n), \mathbb{E}^n)$-manifold is called a *Euclidean similarity n-manifold*.

If $\Gamma < I(\mathbb{H}^n)$ is a discrete subgroup acting freely and properly discontinuously on $\mathbb{H}^n$, then the quotient map $\pi : \mathbb{H}^n \to \mathbb{H}^n/\Gamma$ is a local isometry and a covering projection. The quotient $\mathbb{H}^n/\Gamma$ is a complete $(I(\mathbb{H}^n), \mathbb{H}^n)$-manifold, with the Riemannian metric inherited from $\mathbb{H}^n$ via the covering projection.

Knots and knot complements

Definition

A *knot* is a smooth embedding $K : S^1 \hookrightarrow S^3$ considered up to ambient isotopy. Equivalently, it is a connected, oriented, one-dimensional submanifold of S^3 diffeomorphic to the circle.

Prime knots can be sorted into 3 disjoint groups based on the characteristics of their complements: Torus knots, satellite knots, and hyperbolic knots, of which hyperbolic knots are by far the most prevalent.

Definition

A knot $K \subset S^3$ is called hyperbolic if its complement $S^3 \setminus K$ admits a complete Riemannian metric of constant sectional curvature -1; that is, the complement has a complete hyperbolic structure.

Knots and knot complements

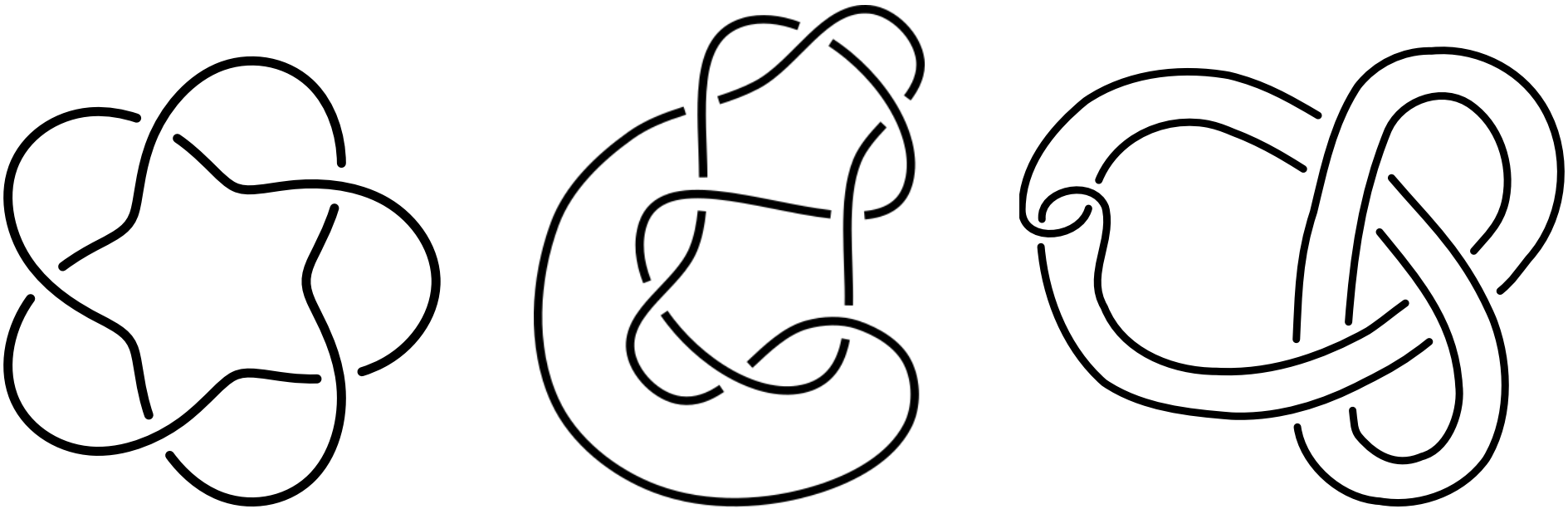


Figure: An example of a torus, hyperbolic and satellite knot respectively

Knots and knot complements

A common problem in knot theory is determining whether two knots are equivalent. By Mostow's Rigidity it's not difficult to see that equivalent knots have equal volume.

Non-equivalent knot complements can have volumes that are arbitrarily close to each other, although this is relatively uncommon. For most non-equivalent knots, knowing the volume of their complement is sufficient to tell them apart.

This motivates the studying of their hyperbolic structures.

Ideal tetrahedra

An ideal tetrahedron in hyperbolic n-space is one with all vertices lying in ∂H^n. In H^3 they can be described up to isometry by just one complex number.

Theorem

Using the half-space model for H^3, for any ideal tetrahedron with vertices $v_1, v_2, v_3, v_4 \in \partial H^3$ there is precisely one orientation preserving isometry sending v_1, v_2, v_3, v_4 to $0, 1, \infty, z$ respectively, with $\Im(z) > 0$ and one orientation reversing isometry sending v_1, v_2, v_3, v_4 respectively to $0, 1, \infty, \bar{z}$.

Definition

Let $\triangle(u, v, w)$ be a Euclidean triangle in with vertices labeled counter-clockwise. The vertex invariants associated with each vertex are gives by the ratio of the adjacent sides:

$$z(u) = \frac{w-u}{v-u} \quad z(v) = \frac{u-v}{w-v} \quad z(w) = \frac{v-w}{u-w}.$$

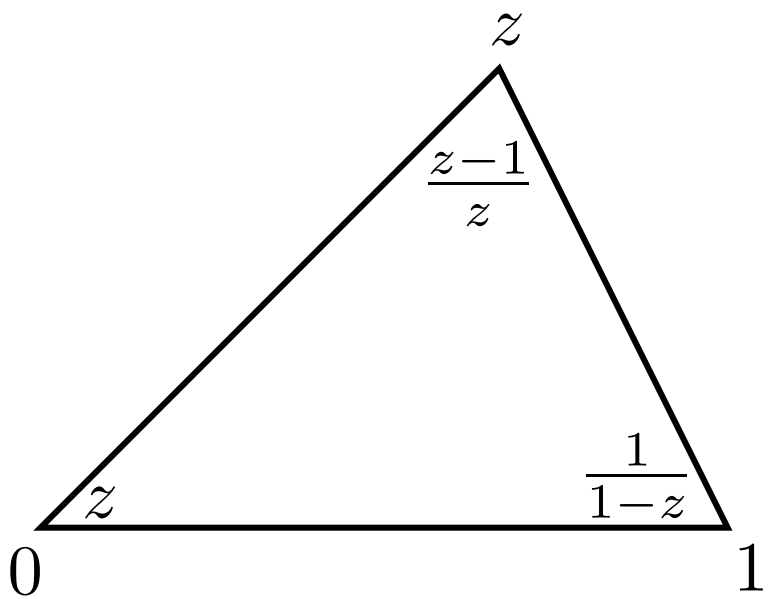


Figure: Vertex invariants of a triangle with vertices $0, 1, z$

Gluing

Definition (G-Side-Pairing)

A *G-side-pairing* for $\mathcal{P}$ is a subset of G: $\Phi = \{g_S : S \in \mathcal{S}\}$ for which the following hold:

- For each side $S \in \mathcal{S}$, there is a unique side $S' \in \mathcal{S}$ with $g_S(S') = S$ and $g_{S'} = g_S^{-1}$
- $P \cap g_S(P') = S$ for sides $S \subset P$, $S' \subset P'$

The equivalence classes of Φ are called *cycles*.

Definition

The *normalized solid angle* of P at a point x of its side S is the real number

$$\omega(P, x) = \frac{\mathrm{Vol}(P \cap B(x, r))}{\mathrm{Vol}(B(x, r))}$$

Gluing

Definition

Let $[x] = \{x_1, ..., x_m\}$ be a cycle and $P_1, ..., P_m \in \mathcal{P}$ be polyhedra containing the points $x_1, ..., x_m$ respectively. The *normalized solid angle sum* of $[x]$ is defined as:

$$\omega[x] = \omega(P_1, x_1) + ... + \omega(P_m, x_m).$$

Definition (Proper Side-Pairing)

Φ is *proper* if each cycle $[x]$ is finite and has normalized solid angle sum $\omega[x] = 1$.

Gluing

Theorem (Gluing 3-manifolds)

Let Φ be a proper G-side pairing for $\mathcal{P}$. Then:

1. *g_S fixes no point of S' for each $S \in \mathcal{S}$*
2. *the sides S and S' are equal if and only if S is a great 2-sphere of S^3 and g_S is the antipodal map of S^3*
3. *Each point of an edge of a polyhedron $P \in \mathcal{P}$ is contained in a cycle at most once*
4. *The quotient space consisting of cycles of Φ is a (G, X)-manifold.*

An analogous theorem can be formulated for 2-manifolds.

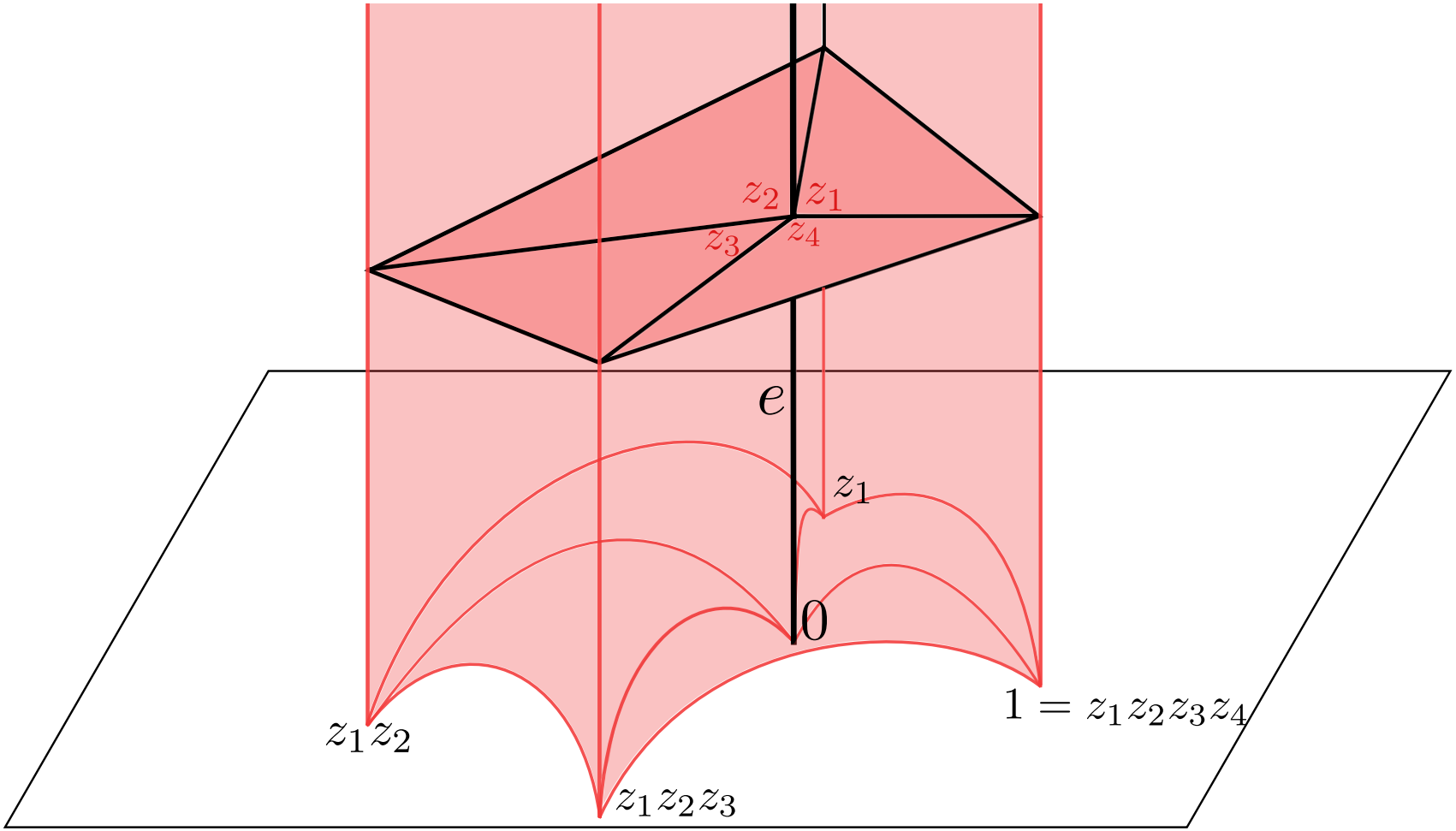
Consistency equations

Theorem (Consistency equations)

Let Φ be an $I(H^3)$ side-pairing for a finite collection $\mathcal{T}$ of k disjoint ideal tetrahedra in H^3. If the set $\mathcal{E}$, of edges of the tetrahedra in $\mathcal{T}$ is subdivided into k cycles of edges and the invariants of each edge cycle $\{e_1, \ldots, e_m\}$ satisfy the equation

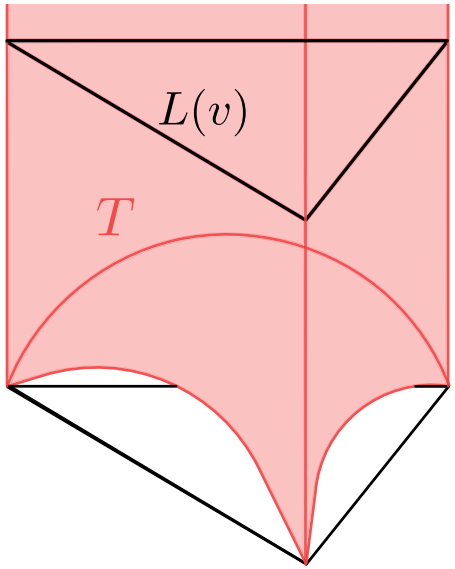
$$z(e_1)z(e_2)...z(e_m) = 1,$$

then Φ is proper.


z_2
z_1
z_3
z_4
e
z_1
0
$z_1 z_2$
$z_1 z_2 z_3$
$1 = z_1 z_2 z_3 z_4$

Cusps and links

In order to get a complete hyperbolic manifold, we need to ensure that the gluing is "well behaved" near the ideal vertices. We call the vertex cycles $[v] = \{v_1, ..., v_k\}$ *cusp points* and we define the link of a cusp point $L[v]$ as the space obtained by gluing together the triangles $L(v)$.



Theorem

Let M be an orientable hyperbolic 3-manifold obtained by gluing ideal polyhedra. Then the link of each cusp $L[v]$ is a torus.

Completeness equations

A link $L[v]$ is complete if the holonomy $\mu : \pi_1(L[v]) \to S_0(\mathbb{E}^2)$ maps $\pi_1(L[v])$ isomorphically onto a freely acting discrete group of Euclidean isometries of $\mathbb{E}^2$.

By identifying $\mathbb{E}^2$ with $\mathbb{C}$ we can see that this is the case if and only if each element of $S_0(E^2) \cong S_0(\mathbb{C})$ in the image of μ is of the form $\varphi(z) = z + \alpha$ with $\alpha \in \mathbb{C} - \{0\}$, hence $\varphi'(z) = 1$.

We fix for every torus $L[v]$ two generators a, b of $\pi_1(L[v])$. If M is to be complete, we must have $\mu'(a) = 1$ and $\mu'(b) = 1$, so we get two equations for each cusp point.

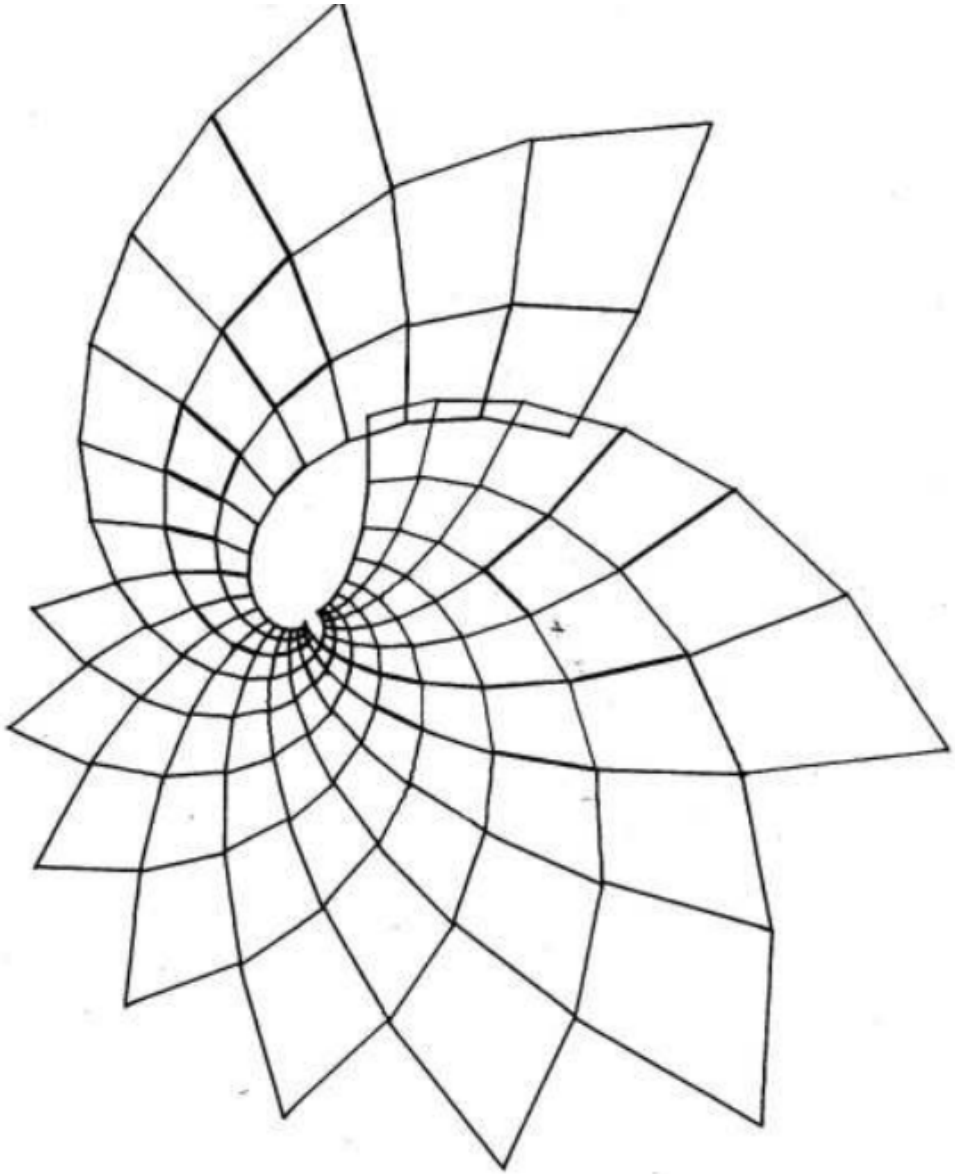


Figure: Example of a non-complete link $L[v]$ developed onto from Thurston's lecture notes

Figure-8 knot

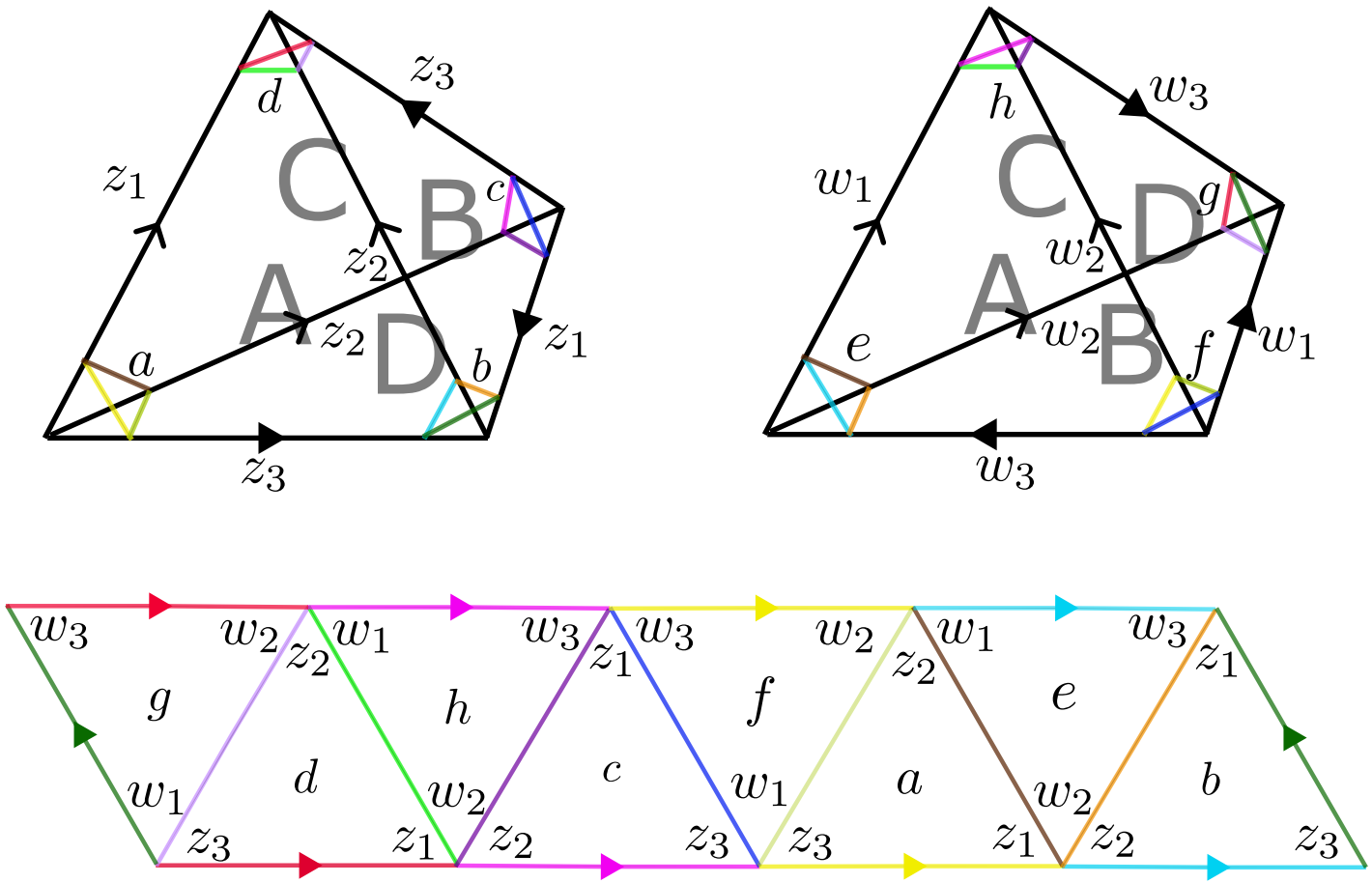


Figure: L developed onto $\mathbb{C}$

The 8_3-knot complement

The previous example is too special and does not give much insight into what happens in a more general case. We may attempt to analyze some other examples. This is where software such as Regina come in handy.

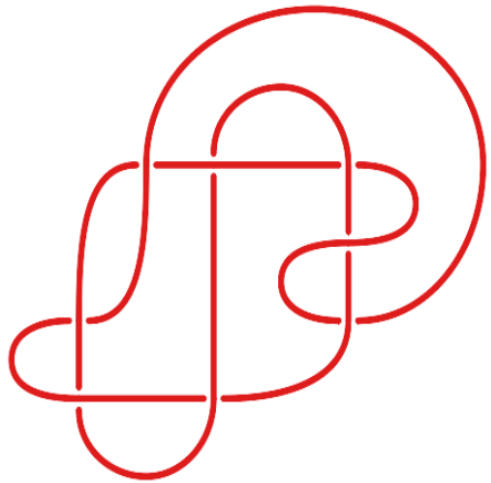


Figure: The 8_3 knot

To make things easier, we will be working on a non-orientable manifold to obtain the consistency equations, and triangulate the 8_3 knot complement using isometric pairs of the 3 tetrahedra.

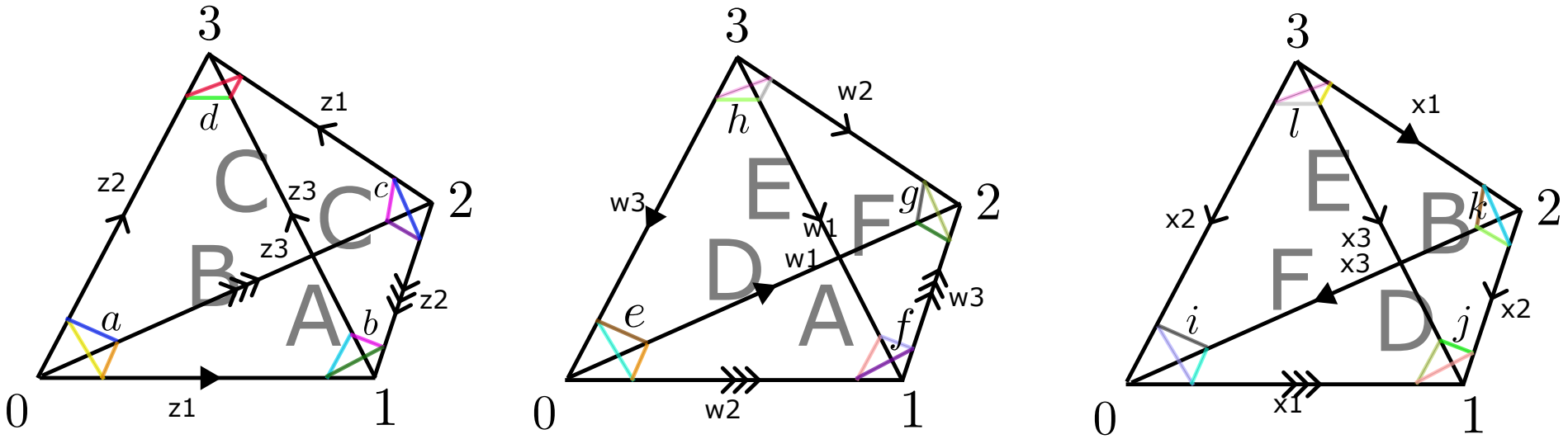


Figure: Gluing pattern for the non-orientable manifold double covered by the 8_3-knot complement

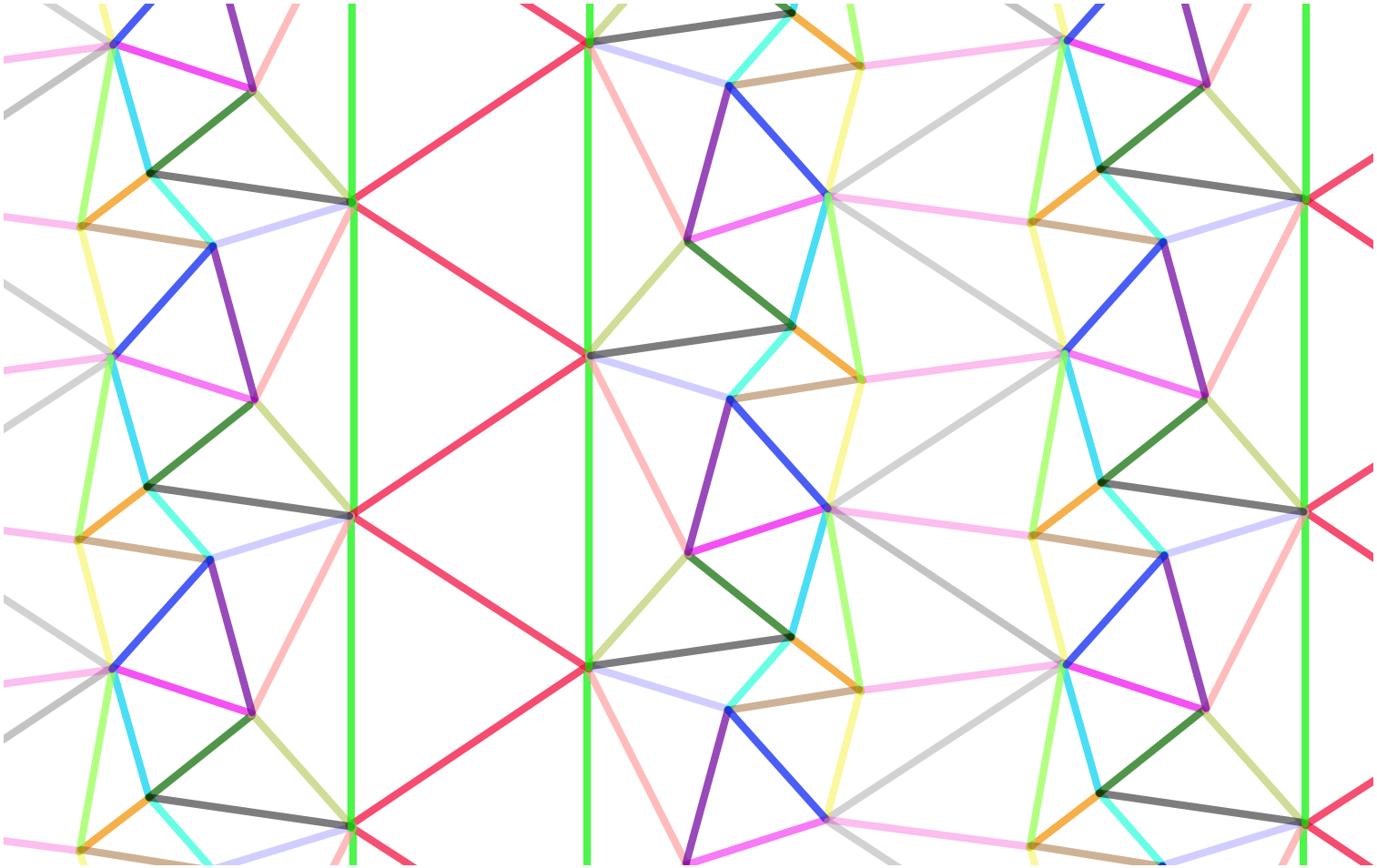


Figure: $L[v]$ developed onto $\mathbb{C}$

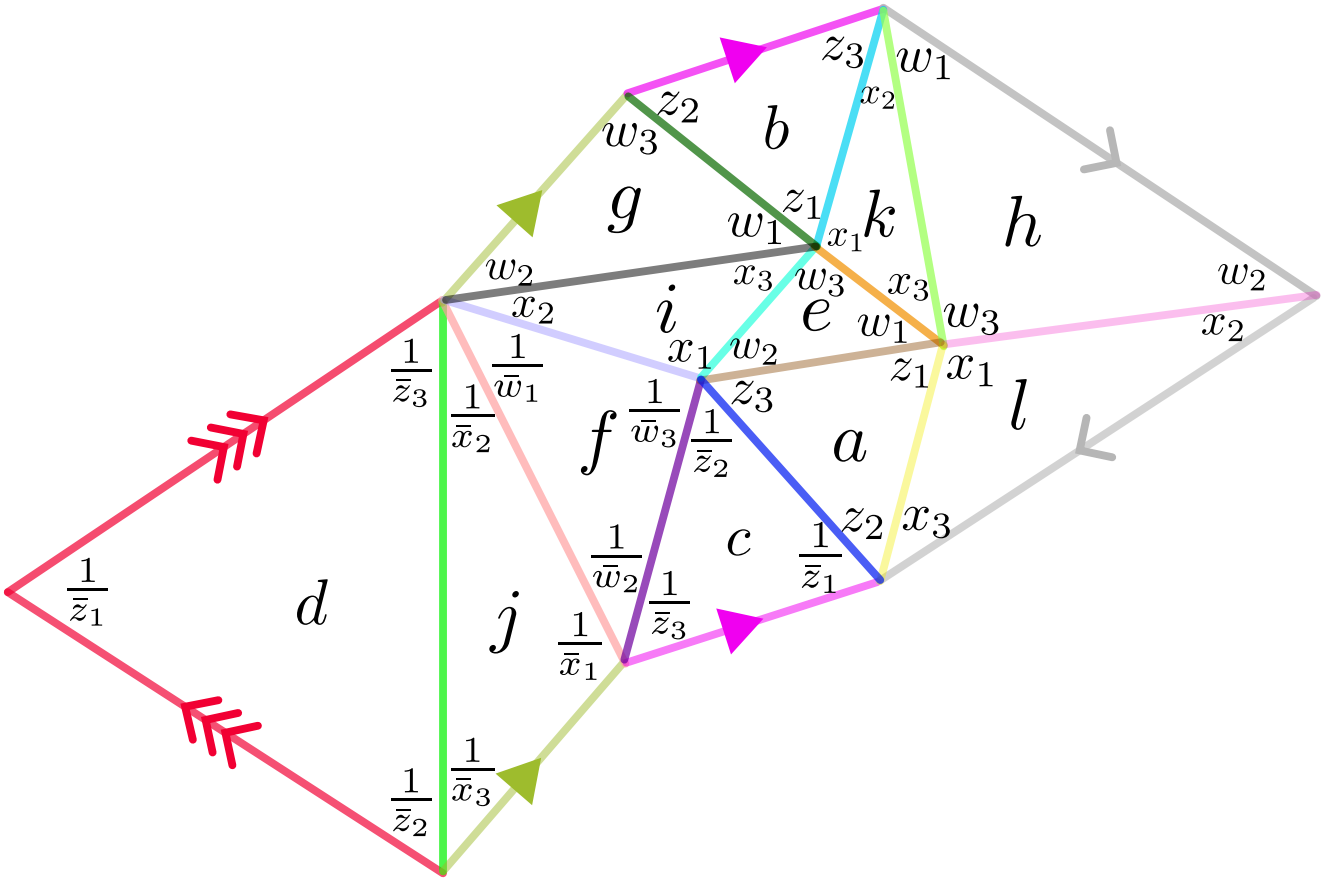

z_3 w_1 x_2
z_2 w_3 b
g z_1 w_1 x_1 k h
w_2 x_3 w_3 x_3 w_2
x_2 i e w_1 w_3 x_2
x_1 w_2 z_1 x_1
$\frac{1}{\bar{z}_3}$ $\frac{1}{\bar{w}_1}$ z_3
$\frac{1}{\bar{x}_2}$ f $\frac{1}{\bar{w}_3}$ $\frac{1}{\bar{z}_2}$ a l
$\frac{1}{\bar{w}_2}$ c z_2 x_3
$\frac{1}{\bar{z}_1}$ d j $\frac{1}{\bar{z}_3}$ $\frac{1}{\bar{z}_1}$
$\frac{1}{\bar{x}_1}$
$\frac{1}{\bar{z}_2}$ $\frac{1}{\bar{x}_3}$

Figure: Fundamental polygon for the Klein bottle link $L[v]$

Consistency equations

$$w_2 x_2 \frac{1}{\bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{w}_1 \bar{x}_2 \bar{x}_3} = 1$$

$$z_1 w_1 w_3 x_1 x_3 = 1$$

$$x_1 w_2 z_3 \frac{1}{\bar{z}_2 \bar{w}_3} = 1.$$

After simplifying, we get

$$z = \frac{x}{w} = \frac{1}{(x-1)(w-1)}.$$

Any potential solution is of the form

$$(z, w, x) = \left(\frac{x^2 - x - 1}{x - 1}, \ \frac{x^2 - x}{x^2 - x - 1}, \ x \right).$$

Completeness

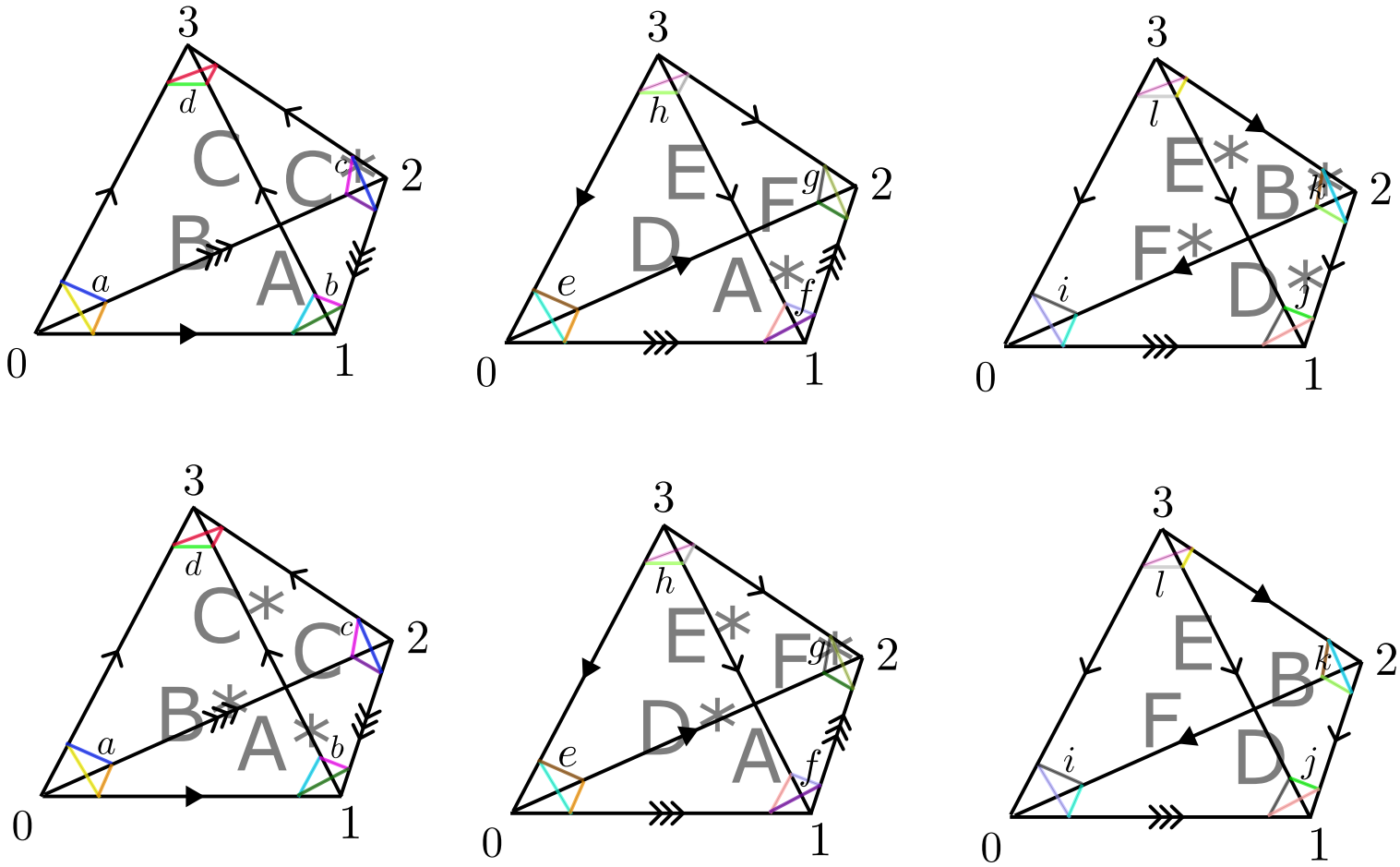
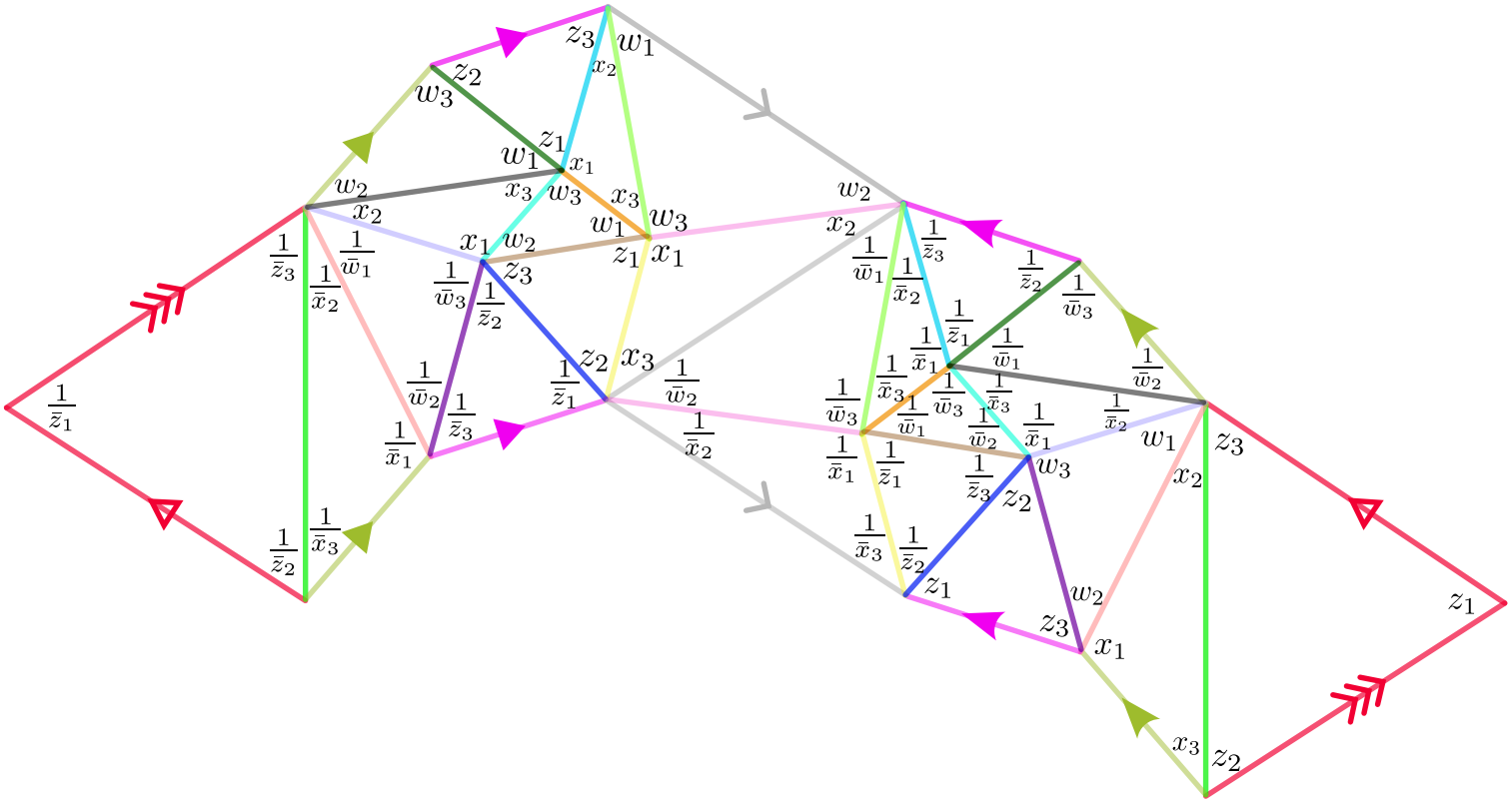


Figure: Gluing pattern for the 8_3 knot complement



Completeness equations:

$$\bar{z}_1 z_2 z_3 = -1,$$

$$\frac{z_2 x_3 x_2 w_2 z_1 z_3 w_2 x_1 x_3}{\bar{x}_3 \bar{x}_1 \bar{w}_2 \bar{z}_3 \bar{z}_1 \bar{x}_3 \bar{z}_2} = 1.$$

Completeness

After simplifying the second equation we get

$$\frac{(x-1)\overline{x}\,\overline{(1-w)}}{\overline{(x-1)^2}\,x(1-w)^2} = 1.$$

So we are looking for points on the curve
$(z, w, x) = \left(\frac{x^2-x-1}{x-1},\ \frac{x^2-x}{x^2-x-1},\ x\right)$ that satisfy the above equation.
Numerical solution using SciPy's fsolve:

$$z \approx 0.39656 + 0.91801i,$$

$$w \approx 0.45563 + 0.47052i,$$

$$x \approx -0.25126 + 0.60486i.$$

Volume

Theorem

The volume of an ideal tetrahedron in H^3 with dihedral angles α, β, γ is given by:

$$Vol(T) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where Λ is the Lobachevsky function

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|\, dt.$$

This makes it easy to approximate the volumes of triangulated manifolds, given the dihedral angles.

$$Vol(8_3\text{-knot complement}) \approx 2(1.0094 + 0.8935 + 0.7165) \approx 5.2387$$

UiO : Department of Mathematics
University of Oslo

János Apró

Hyperbolic Structure of Knot Complements